TAME AUTOMORPHISMS WITH MULTIDEGREES IN THE FORM OF ARITHMETIC PROGRESSIONS

JIANTAO LI AND XIANKUN DU

Abstract. Let $(a, a+d, a+2d)$ be an arithmetic progression of positive integers. The following statements are proved:
(1) If $a \mid 2d$, then $(a, a+d, a+2d) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.
(2) If $a \nmid 2d$, then, except for arithmetic progressions of the form $(4i, 4i+ij, 4i+2ij)$ with $i, j \in \mathbb{N}$ and j is an odd number, $(a, a+d, a+2d) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$. We also related the exceptional unknown case to a conjecture of Jie-tai Yu, which concerns with the lower bound of the degree of the Poisson bracket of two polynomials.

1. Introduction

Throughout this paper, let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map on $\mathbb{C}^n$, where $\mathbb{C}$ denotes the complex field. Denote by $\mathrm{mdeg}F := (\deg F_1, \ldots, \deg F_n)$ the *multidegree* of F. Denote by $\mathrm{Aut}(\mathbb{C}^n)$ the group of all polynomial automorphisms of $\mathbb{C}^n$ and by mdeg the mapping from the set of all polynomial maps into the set $\mathbb{N}^n$, here and throughout, $\mathbb{N}$ denotes the set of all positive integers.

A polynomial automorphism $F = (F_1, \ldots, F_n)$ of $\mathbb{C}^n$ is called *elementary* if

$$F = (x_1, \ldots, x_{i-1}, \alpha x_i + f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), x_{i+1}, \ldots, x_n)$$

for $\alpha \in \mathbb{C}^*$. Denote by $\mathrm{Tame}(\mathbb{C}^n)$ the subgroup of $\mathrm{Aut}(\mathbb{C}^n)$ that is generated by all elementary automorphisms. The elements of $\mathrm{Tame}(\mathbb{C}^n)$ are called *tame automorphisms*. The classical Jung-van der Kulk theorem [3, 4] showed that every polynomial automorphism of $\mathbb{C}^2$ is tame. For many years people believe that $\mathrm{Aut}(\mathbb{C}^n)$ is equal to $\mathrm{Tame}(\mathbb{C}^n)$. However, in 2004, Shestakov and Umirbaev [11, 12] proved the famous Nagata conjecture, that is, the Nagata automorphism on $\mathbb{C}^3$ is not tame.

The multidegree plays an important role in the description of polynomial automorphisms. It follows from Jung-van der Kulk theorem that if $F = (F_1, F_2) \in \mathrm{Aut}(\mathbb{C}^2)$ then $\mathrm{mdeg}F = (\deg F_1, \deg F_2)$ is principal, that is, either $\deg F_1 | \deg F_2$ or $\deg F_2 | \deg F_1$. And the famous Jacobian conjecture is equivalent to the assert that if (F_1, F_2) is a polynomial map satisfying the Jacobian condition, then $\mathrm{mdeg}F = (\deg F_1, \deg F_2)$ is principal [1]. But it is difficult to describe the multidegrees of polynomial maps in higher dimensions, even in dimension three. Recently, Karaś presented a series of papers concerned with multidegrees of tame automorphisms in dimension three. It is shown in [5, 6] that there is no tame automorphism

Date: December 30, 2011.
2010 *Mathematics Subject Classification.* 14R10.
Key words and phrases. multidegree, tame automorphism, elementary reduction, arithmetic progression.
This research was partially supported by NSF of China (No.11071097, No.11101176) and "211 Project" and "985 Project" of Jilin University.

of $\mathbb{C}^3$ with multidegree $(3,4,5)$ and $(4,5,6)$. In [7], it is proved that $(p_1, p_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in p_1\mathbb{N} + p_2\mathbb{N}$, where $2 < p_1 < p_2$ are prime numbers. In [8, 9], similar conclusions are given: $(3, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ if and only if $3|d_2$ or $d_3 \in 3\mathbb{N} + d_2\mathbb{N}$; $(d_1, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$, where d_1, d_2 are coprime odd numbers.

Let $(a, a+d, a+2d)$ be an arithmetic progression of positive integers. The following statements are proved in this paper.

(1) If $a \mid 2d$, then $(a, a+d, a+2d) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$.
(2) If $a \nmid 2d$, then $(a, a+d, a+2d) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ with the exceptional unknown case of the form $(4i, 4i+ij, 4i+2ij)$ with $i, j \in \mathbb{N}$ and j is an odd number. We also related this unknown case to a conjecture of Jie-tai Yu, which concerns with the lower bound of the degree of the Poisson bracket of two polynomials.

2. Preliminaries

Recall that a pair $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ is called $*$-*reduced* in [11, 12] if

(1) f, g are algebraically independent;
(2) $\bar{f}, \bar{g}$ are algebraically dependent, where $\bar{f}$ denotes the highest homogeneous component of f;
(3) $\bar{f} \notin \langle \bar{g} \rangle$ and $\bar{g} \notin \langle \bar{f} \rangle$.

The following inequality plays an important role in the proof of the Nagata conjecture in [11, 12] and is also essential in our proofs.

Theorem 2.1. ([11, Theorem 3]). *Let $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ be a $*$-reduced pair, and $G(x,y) \in \mathbb{C}[x,y]$ with $\deg_y G(x,y) = pq + r$, $0 \le r < p$, where $p = \frac{\deg f}{\gcd(\deg f, \deg g)}$. Then*

$$\deg G(f,g) \ge q(p \deg g - \deg f - \deg g + \deg[f,g]) + r \deg g.$$

Note that $[f, g]$ means the Poisson bracket of f and g:

$$[f,g] = \sum_{1 \le i < j \le n} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)[x_i, x_j].$$

By definition $\deg[x_i, x_j] = 2$ for $i \ne j$ and $\deg 0 = -\infty$,

$$\deg[f,g] = \max_{1 \le i < j \le n} \deg\left\{\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)[x_i, x_j]\right\}.$$

It is shown in [11] that $[f, g] = 0$ if and only if f, g are algebraically dependent. And if f, g are algebraically independent, then

$$\deg[f,g] = 2 + \max_{1 \le i < j \le n} \deg\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)$$

Remark 2.2. It is easy to shown (see [5] for example) that Theorem 2.1 is true even if f, g just satisfy: (1) f, g are algebraically independent; (2) $\bar{f} \notin \langle \bar{g} \rangle$ and $\bar{g} \notin \langle \bar{f} \rangle$.

Theorem 2.3. ([12, Theorem 2]). *Let $F = (F_1, F_2, F_3)$ be a tame automorphism of $\mathbb{C}^3$. If $\deg F_1 + \deg F_2 + \deg F_3 > 3$, (that is, F is not a linear automorphism), then F admits either an elementary reduction or a reduction of types I-IV (see* [12, Definitions 1-4]*).*

Recall that we say a polynomial automorphism $F = (F_1, F_2, F_3)$ admits an *elementary reduction* if there exists a polynomial $g \in \mathbb{C}[x, y]$ and a permutation σ of the set $\{1, 2, 3\}$ such that $\deg(F_{\sigma(1)} - g(F_{\sigma(2)}, F_{\sigma(3)})) < \deg F_{\sigma(1)}$.

3. Main results

Note that if (F_1, F_2, F_3) with multidegree (d_1, d_2, d_3) is a tame automorphism, then, after a permutation σ, $(F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)}) \in \mathrm{Tame}(\mathbb{C}^3)$. Thus, without loss of generality we can assume that $d_1 \le d_2 \le d_3$. Next, by [5, Proposition 2.2] it follows that if $d_1 | d_2$ or d_3 is a linear combination of d_1 and d_2 with coefficients in $\mathbb{N}$, then $(d_1, d_2, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.

The task now is to show when does an arithmetic progression $(a, a + d, a + 2d)$ belong to $\mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$, where $a, d \in \mathbb{N}$.

Lemma 3.1. *An arithmetic progression $(a, a+d, a+2d)$ satisfies $a \mid d$ or $a+2d \in a\mathbb{N} + (a + d)\mathbb{N}$ if and only if $a \mid 2d$.*

Proof. If $a + 2d \in a\mathbb{N} + (a + d)\mathbb{N}$, then there exist $i, j \in \mathbb{N}$ such that $a + 2d = ai + (a + d)j$, whence $(i + j - 1)a = (2 - j)d$.

(1) If $i + j > 1$. Thus, $2 - j > 0$, $j = 0$ or 1. If $j = 0$, then $2d = (i - 1)a$. If $j = 1$, then $d = ia$. In both case we get $a \mid 2d$.
(2) Otherwise, if $i + j \le 1$, it is easy to see that $d = 0$ and hence $a \mid 2d$. More precisely, if $i = j = 0$, then $a = -2d$, whence $d = 0$; if $i = 0$, $j = 1$ or $i = 1$, $j = 0$, it is trivial that $d = 0$.

Conversely, suppose that $a \mid 2d$. If a is odd, then $a \mid d$. If a is even, it follows from $a \mid 2d$ that $d = \frac{a}{2}m$ for some $m \in \mathbb{N}$, whence $a + 2d = (m + 1)a \in a\mathbb{N}$. $\square$

Lemma 3.2. *To prove that $(a, a + d, a + 2d) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$, it is enough to show that every polynomial automorphism F with multidegree $(a, a+d, a+2d)$ does not admit any elementary reduction.*

Proof. By [10, Theorem 27], to prove that there is no tame automorphism of $\mathbb{C}^3$ with multidegree (d_1, d_2, d_3), it suffices to show that such a hypothetical automorphism admits neither an elementary reduction nor a reduction of type III.

If F with multidegree $(a, a + d, a + 2d)$ admits a reduction of type III, then by [12, Definition 3] there exists $n \in \mathbb{N}$ such that

$$(1)\begin{cases} n < a \le \dfrac{3}{2}n, \\ a + d = 2n, \\ a + 2d = 3n, \end{cases} \quad \text{or } (2)\begin{cases} a = \dfrac{3}{2}n, \\ a + d = 2n, \\ 5n/2 < a + 2d \le 3n. \end{cases}$$

It follows from the last two equalities in (1) that $a = d = n$, which contradicts $n < a \le \frac{3}{2}n$. It follows from the first two equalities in (2) that $a = \frac{3}{2}n, d = \frac{1}{2}n$, which contradicts $5n/2 < a + 2d \le 3n$. Therefore, F admits no reduction of type III. Thus, to prove that $(a, a + d, a + 2d) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$, it is enough to show that every polynomial automorphism F with multidegree $(a, a+d, a+2d)$ does not admit any elementary reduction. $\square$

We are now in a position to show our main results.

Theorem 3.3. *Let $(a, a+d, a+2d)$ be an arithmetic progression of positive integers.*

(1) *If $a \mid 2d$, then $(a, a + d, a + 2d) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.*

(2) *If* $a \nmid 2d$, *then* $(a, a+d, a+2d) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$, *except for the case that* $(4i, 4i+ij, 4i+2ij)$ *with* $i, j \in \mathbb{N}$ *and* j *is an odd number.*

Proof. (1) If $a \mid 2d$, then $a \mid d$ or $a+2d \in a\mathbb{N} + (a+d)\mathbb{N}$ by Lemma 3.1, and hence $(a, a+d, a+2d) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ by [5, Proposition 2.2].

(2) Now suppose that $a \nmid 2d$. Let $F = (F_1, F_2, F_3)$ be a polynomial automorphism with multidegree $(a, a+d, a+2d)$. Then by Lemma 3.2, it suffices to show that F admits no elementary reduction. By Lemma 3.1, the condition $a \nmid 2d$ implies that $a \nmid d$ and $a+2d \notin a\mathbb{N} + (a+d)\mathbb{N}$.

Set $\gcd(a, a+d) = \gcd(a+d, a+2d) = \gcd(a, d) = b$. Write $a = b\bar{a}, d = b\bar{d}$. Then $\bar{d} \geq 1$ and we can claim that $\bar{a} \geq 3$, since

(i) if a is an odd number, it follows from $a \nmid 2d$ that $b \neq a$, whence $\bar{a} = \frac{a}{b} \geq 3$.
(ii) if a is an even number, it follows from $a \nmid 2d$ that $b \neq a$ and $b \neq \frac{a}{2}$, whence $\bar{a} \geq 3$.

Now the proof proceeds into three cases.

Case 1: If F admits an elementary reduction of the form $(F_1, F_2, F_3 - g(F_1, F_2))$ such that $\deg(F_3 - g(F_1, F_2)) < \deg F_3$, then $\deg F_3 = \deg g(F_1, F_2)$. Since $\gcd(a, a+d) = b$, we have $p = \frac{\deg F_1}{\gcd(\deg F_1, \deg F_2)} = \bar{a}$. Set $\deg_y g(x, y) = \bar{a}q + r,\ 0 \leq r < \bar{a}$. Since (F_1, F_2, F_3) is a polynomial automorphism, it follows that $F_i, F_j (i, j = 1, 2, 3)$ are algebraically independent, and hence $\deg[F_i, F_j] \geq 2$. Moreover, $\bar{F_i} \notin \langle \bar{F_j} \rangle$ since otherwise we have $\deg F_i | \deg F_j$ that contradicts to the fact that $a \nmid (a+d)$ and $a+2d \notin a\mathbb{N} + (a+d)\mathbb{N}$. By Remark 2.2,

$$\begin{aligned} a+2d = \deg F_3 &= \deg g(F_1, F_2) \\ &\geq q(\bar{a}(a+d) - a - (a+d) + \deg[F_1, F_2]) + r(a+d) \\ &\geq q(3(a+d) - a - (a+d) + 2) + r(a+d) \\ &= q(a+2d+2) + r(a+d). \end{aligned}$$

Thus, $q = 0$ and $r \leq 1$. Write $g(F_1, F_2) = g_1(F_1) + g_2(F_1)F_2$. Since $a\mathbb{N} \cap (a\mathbb{N} + (a+d)) = \emptyset$, we have $a+2d = \deg F_3 = \deg g(F_1, F_2) \in a\mathbb{N}$ or $a\mathbb{N} + (a+d)$, which contradicts $a+2d \notin a\mathbb{N} + (a+d)\mathbb{N}$.

Case 2: If F admits an elementary reduction of the form $(F_1 - g(F_2, F_3), F_2, F_3)$, then $\deg F_1 = \deg g(F_2, F_3)$. Since $\gcd(a+d, a+2d) = b$, we have $p = \bar{a} + \bar{d}$. Set $\deg_y g(x, y) = (\bar{a} + \bar{d})q + r,\ 0 \leq r < \bar{a} + \bar{d}$. Then

$$\begin{aligned} a = \deg F_1 &= \deg g(F_2, F_3) \\ &\geq q((\bar{a} + \bar{d})(a+2d) - (a+d) - (a+2d) + \deg[F_2, F_3]) + r(a+2d) \\ &\geq q(4(a+2d) - (a+d) - (a+2d) + 2) + r(a+2d) \\ &= q(2a+5d+2) + r(a+2d). \end{aligned}$$

Thus, $q = r = 0$. Write $g(F_2, F_3) = g_1(F_2)$. Then $a = \deg F_1 = \deg g_1(F_2) \in (a+d)\mathbb{N}$, contrary to $a < a+d$.

Case 3: If F admits an elementary reduction of the form $(F_1, F_2 - g(F_1, F_3), F_3)$, then $\deg F_2 = \deg g(F_1, F_3)$.

(i) If a is an odd number, then $\gcd(a, a+2d) = \gcd(a, d)$ and $p = \bar{a}$. Set $\deg_y g(x, y) = \bar{a}q + r,\ 0 \le r < \bar{a}$. Then

$$\begin{aligned} a + d = \deg F_2 &= \deg g(F_1, F_3) \\ &\ge q(\bar{a}(a+2d) - a - (a+2d) + \deg[F_1, F_3]) + r(a+2d) \\ &\ge q(3(a+2d) - a - (a+2d) + 2) + r(a+2d) \\ &= q(a+4d+2) + r(a+2d). \end{aligned}$$

Thus, $q = r = 0$. Write $g(F_1, F_3) = g_1(F_1)$, which implies that $a + d = \deg F_2 = \deg g(F_1, F_3) = \deg g_1(F_1) \in a\mathbb{N}$, contrary to $a \nmid a + d$.

(ii) Now let a be an even number. If we additionally assume that $a \nmid 4d$, then $\gcd(a, a+2d) = \gcd(a, 2d) = 2\gcd(\frac{a}{2}, d) \neq a, \frac{a}{2}$, whence $p = \frac{a}{\gcd(a,a+2d)} \neq 1, 2$. Therefore $p \ge 3$. Set $\deg_y g(x, y) = pq + r,\ 0 \le r < p$. Then

$$\begin{aligned} a + d = \deg F_2 &= \deg g(F_1, F_3) \\ &\ge q(p(a+2d) - a - (a+2d) + \deg[F_1, F_3]) + r(a+2d) \\ &\ge q(3(a+2d) - a - (a+2d) + 2) + r(a+2d) \\ &= q(a+4d+2) + r(a+2d). \end{aligned}$$

Thus, $q = r = 0$. A same contradiction follows as in (a).
Moreover, if a is an even number and $4 \nmid a$, then the condition $a \nmid 2d$ forces $a \nmid 4d$. More precisely, if a is even with $4 \nmid a$, then $a = 2k$ for some odd number k. If $a \mid 4d$, then $k \mid 2d$ and hence $k \mid d$. Thus, $a \mid 2d$, a contradiction. Therefore, the only unknown case left is $4 \mid a$ and $a \mid 4d$, that is, $(4i, 4i+ij, 4i+2ij)$ with $i, j \in \mathbb{N}$. Moreover, the condition $a \nmid 2d$ forces j to be an odd number.

Thus, except for the case that $(4i, 4i+ij, 4i+2ij)$ with $i, j \in \mathbb{N}$ and j is an odd number, any polynomial automorphism F with multidegree $(a, a+d, a+2d)$ admits no elementary reduction, and consequently, $(a, a+d, a+2d) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$. $\square$

Now the only unknown case left is to show whether there is a tame automorphism with multidegree $(4i, 4i+ij, 4i+2ij)(i, j \in \mathbb{N}$ and j is an odd number). By Lemma 3.2, to show such an automorphism does not exist, we just need to show it admits no elementary reduction.

Theorem 3.4. *Let $F = (F_1, F_2, F_3)$ be a polynomial automorphism with multidegree $(4i, 4i+ij, 4i+2ij)$ that $i, j \in \mathbb{N}$ and j is an odd number. If $\deg[F_1, F_3] > \deg F_1$, then F admits no elementary reductions.*

Proof. Proceeding as in the proof of Theorem 3.3, the proof falls into three parts.

(1) If F admits an elementary reduction of the form $(F_1, F_2, F_3 - g(F_1, F_2))$, then $\deg F_3 = \deg g(F_1, F_2)$. Since $\gcd(4i, 4i+ij) = i\gcd(4, j) = i$, we have $p = \frac{\deg F_1}{\gcd(\deg F_1, \deg F_2)} = 4$. Set $\deg_y g(x, y) = 4q + r,\ 0 \le r < 4$. Then

$$\begin{aligned} 4i + 2ij = \deg F_3 &= \deg g(F_1, F_2) \\ &\ge q(4(4i+ij) - 4i - (4i+ij) + \deg[F_1, F_2]) + r(4i+ij) \\ &\ge q(8i+3ij+2) + r(4i+ij). \end{aligned}$$

Thus, $q = 0$ and $r \le 1$. Write $g(F_1, F_2) = g_1(F_1) + g_2(F_1)F_2$. However, the conditions $4i\mathbb{N} \cap ((4i+ij) + 4i\mathbb{N}) = \emptyset$ and $4i + 2ij \notin 4i\mathbb{N} + (4i+ij)\mathbb{N}$ imply that $\deg F_3 = \deg g(F_1, F_2)$ is impossible.

(2) If F admits an elementary reduction of the form $(F_1 - g(F_2, F_3), F_2, F_3)$, then $\deg F_1 = \deg g(F_2, F_3)$. Since $\gcd(4i + ij, 4i + 2ij) = i \gcd(4, j) = i$, $p = 4 + j$. Set $\deg_y g(x, y) = (4 + j)q + r,\ 0 \le r < 4 + j$. Then

$$\begin{aligned} 4i = \deg F_1 &= \deg g(F_2, F_3) \\ &\ge q((4 + j)(4i + 2ij) - (4i + ij) - (4i + 2ij) + \deg[F_2, F_3]) + r(4i + 2ij) \\ &\ge q(8i + 9ij + 2ij^2 + 2) + r(4i + 2ij). \end{aligned}$$

Thus, $q = r = 0$. Write $g(F_2, F_3) = g_1(F_2)$. Then $4i = \deg F_1 = \deg F_2 \in (4i+ij)\mathbb{N}$, a contradiction.

(3) If F admits an elementary reduction of the form $(F_1, F_2 - g(F_1, F_3), F_3)$, then $\deg F_2 = \deg g(F_1, F_3)$. Since $\gcd(4i, 4i + 2ij) = 2i \gcd(2, j) = 2i$, $p = 2$. Set $\deg_y g(x, y) = 2q + r,\ 0 \le r < 2$. Then

$$\begin{aligned} 4i + ij = \deg F_2 &= \deg g(F_1, F_3) \\ &\ge q(2(4i + 2ij) - 4i - (4i + 2ij) + \deg[F_1, F_3]) + r(4i + 2ij) \\ &> q(2ij + 4i) + r(4i + 2ij). \end{aligned}$$

Note that the last inequality follows from $\deg[F_1, F_3] > \deg F_1$. Thus, $q = r = 0$. Write $g(F_1, F_3) = g_1(F_1)$. Then we get a contradiction to $4i \nmid 4i + ij$.

Consequently, F admits no elementary reductions. □

The task now is to ask what is the lower bound of $\deg[F_1, F_3]$, particularly, if $\deg[F_1, F_3] > \deg F_1$ for all polynomials satisfying $\deg F_1 = 4i$ and $\deg F_3 = 4i+2ij$, $i, j \in \mathbb{N}$ and j is an odd number, then we can give a complete description of whether $(a, a+d, a+2d) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. This question is closely related to a conjecture of Jie-Tai Yu.

Conjecture 3.5. [2] Let f and g be algebraically independent polynomials in $k[x_1, \dots, x_n]$ such that the homogeneous components of maximal degree of f and g are algebraically dependent, f and g generate their integral closures $C(f)$ and $C(g)$ in $k[x_1, \dots, x_n]$, respectively, and neither $\deg f | \deg g$ nor $\deg g | \deg f$, then

$$\deg[f, g] > \min\{\deg(f), \deg(g)\}.$$

Although Conjecture 3.5 has some counterexamples in [2], it is still of great interest to find a meaningful lower bound of $\deg[f, g]$, and such a bound will give a nice description of $\operatorname{Tame}(\mathbb{C}^n)$ and $\operatorname{Aut}(\mathbb{C}^n)$. In particular, if Conjecture 3.5 is valid for f, g with $\deg f = 4i$ and $\deg g = 4i+2ij$, then we can claim that $(a, a+d, a+2d) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $a \nmid 2d$.

Corollary 3.6. (1) *Let (d_1, d_2, d_3) be a sequence of continuous integers, then $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_1 \le 2$.*

(2) *Let (d_1, d_2, d_3) be a sequence of continuous odd numbers, then $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_1 = 1$.*

(3) *Let (d_1, d_2, d_3) be a sequence of continuous odd numbers, then $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if $d_1 \le 4$, $(d_1, d_2, d_3) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if $d_1 > 4$ and $d_1 \ne 8$. The only unknown case left is $(8, 10, 12)$.*

Proof. (1) If $d_1 \le 2$, then $(d_1, d_1 + 1, d_1 + 2) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ by Lemma 3.1. If $d_1 \ge 3$ but $d_1 \ne 4$, then by Theorem 3.3 $(d_1, d_1+1, d_1+2) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. The only case left is $(4, 5, 6)$, which is proved by Karaś [6] that $(4, 5, 6) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$.

(2) It follows from Lemma 3.1 and Theorem 3.3 that a sequence of continuous odd numbers $(d_1, d_2, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ if and only if $d_1 = 1$.

(3) It follows from Lemma 3.1 that $(2k, 2k+2, 2k+4) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ when $k \leq 2$. If $k > 2$ but $k \neq 4$, then by Theorem 3.3 $(2k, 2k+2, 2k+4) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$. The only unknown case left is $(8, 10, 12)$. $\square$

References

1. S. Abhyankar, Some thoughts on the Jacobian Conjecture, Part I, *J. Algebra* **319** (2008) 493–548.
2. V. Drensky, J. T. Yu, Degree estimate for commutators, *J. Algebra* **322**(7) (2009) 2321–2334.
3. H. W. E. Jung, Über ganze birationale Transformationen der Ebene, *J. Reine Angew. Math.* **184** (1942) 161–174.
4. W. van der Kulk, On polynomial rings in two variables, *Nieuw Arch. Wiskd. (3)* **1** (1953) 33–41.
5. M. Karaś, There is no tame automorphism of $\mathbb{C}^3$ with multidegree $(3, 4, 5)$, *Proc. Amer. Math. Soc.* **139**(3) (2011) 769–775.
6. M. Karaś, There is no tame automorphism of $\mathbb{C}^3$ with multidegree $(4, 5, 6)$, arXiv:1104.1061v1 [math.AG] 6 Apr 2011.
7. M. Karaś, Tame automorphisms of $\mathbb{C}^3$ with multidegree of the form (p_1, p_2, d_3), *Bull. Pol. Acad. Sci. Math.* **59** (2011)27–32.
8. M. Karaś, Tame automorphisms of $\mathbb{C}^3$ with multidegree of the form $(3, d_2, d_3)$, *J. Pure Appl. Algebra* **214** (2010) 2144–2147.
9. M. Karaś, J. Zygadło, On multidegrees of tame and wild automorphisms of $\mathbb{C}^3$, *J. Pure Appl. Algebra* **215** (2011) 2843–2846.
10. M. Karaś, Multidegrees of tame automorphisms of $\mathbb{C}^n$, *Dissertationes Math.* **477** (2011), 55 pp. doi:10.4064/dm477-0-1.
11. I. P. Shestakov, U. U. Umirbaev, Poisson brackets and two-generated subalgebras of polynomials, *J. Amer. Math. Soc.* **17** (2004) 181–196.
12. I. P. Shestakov, U. U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, *J. Amer. Math. Soc.* **17** (2004) 197–227.

School of Mathematics, Jilin university, 130012, Changchun, China
E-mail address: jtlimath@gmail.com

School of Mathematics, Jilin university, 130012, Changchun, China
E-mail address: duxk@jlu.edu.cn